

Title	Secondary School Additional Mathematics Material Compilation		
Date	16/6/2019	Editor	Lee Jian Lian

Applicable to the following courses/syllabus ¹

- Singapore-Cambridge GCE 'N'/GCE 'O' Level Additional Mathematics
- Institute of Technical Education (Calculus/Mathematics)
- Nanyang Polytechnic – School of Chemical and Life Sciences – Mathematics for Life Science Module
- Nanyang Polytechnic – School of Engineering – Engineering Mathematics 1A/1B Module
- Nanyang Polytechnic – School of Information Technology – Computing Mathematics Module
- Singapore Cambridge GCE 'A' Level H1/H2 Mathematics ²

Topics	Page Number
Indices and Surds	2
Quadratic Equations – Discriminant, Sum & Product of Roots and Quadratic Inequalities	11
Exponents and Logarithms	17
Binomial Theorem	23
Differentiation – Basic Demonstration of Power Rule, Sum Rule, Difference Rule, Product Rule, Quotient Rule and Chain Rule	27
Differentiation – Involving Logarithmic and Exponential Functions	38
Integration – Basic Demonstration of Constant Multiple Rule, Sum Rule and Difference Rule	42
Integration – Leading to Logarithmic Function and Involving Exponential Functions	49
Definite Integrals – Integration as method to find area under curve	51

¹ The list should not be regarded as exhaustive.

² Both H1 and H2 Mathematics doesn't require specific grades in Additional Mathematics as prerequisite on paper. However, most Junior Colleges will assume a robust knowledge in Additional Mathematics has been acquired before you are permitted to take H2 Mathematics. In Junior Colleges that allow you to take H2 Mathematics with no Additional Mathematics knowledge, this document can provide students some of the basic knowledge required.

Title	Surds and Indices
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	1/6/2019

***This article assumes you understand the Elementary Mathematics topic “Indices” and the techniques used to solve quadratic equations, as many of the concepts in that topic will also be required to understand this topic, nevertheless, revision for those topics will be provided.

Laws of Indices as taught in Elementary Mathematics
$a^m \times a^n = a^{m+n}$
$\frac{a^m}{a^n} = a^{m-n}$
$(a^m)^n = a^{mn}$
$a^{-n} = \frac{1}{a^n}$ (where $a \neq 0$)
$a^0 = 1$ (where $a \neq 0$)
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
$a^{\frac{m}{n}} = \sqrt[n]{a^m}$
$(ab)^n = a^n b^n$

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Basic Surds Rules to Understand before Proceeding

Given that an expression can be written in the following form

$$g\sqrt{ab^2}$$

It can be rewritten as the following

$$gb\sqrt{a}$$

Example 1.1

$\sqrt{8}$ can be decomposed into the following

$$\sqrt{2 \times 4}$$

Since $4 = 2^2$, we can rewrite in the following manner: $2\sqrt{2}$

Law of Surds (Only involving square roots)
$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
$a\sqrt{b} \times c\sqrt{d} = ac(\sqrt{bd})$
$\sqrt{a^2} = a$
$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
$m\sqrt{a} + n\sqrt{a} = \sqrt{a}(m + n)$
$m\sqrt{a} - n\sqrt{a} = \sqrt{a}(m - n)$

Rules of rationalizing the denominator.

Rule 1

If the expression is in the following form

$$\frac{a}{g\sqrt{b}}$$

Multiply by the denominator to both numerator and denominator to get the following

$$\frac{a}{g\sqrt{b}} \times \frac{g\sqrt{b}}{g\sqrt{b}}$$

Rule 2.

If expression is in the following form or show some near resemblance to the following form

$$\frac{a + g\sqrt{b}}{c - d\sqrt{p}}$$

Find the conjugate value of the denominator for which the sign is in between $c - d\sqrt{p}$ is flipped to positive and multiply conjugate value to both numerator and denominator which yields the following result.

$$\frac{a + g\sqrt{b}}{c - d\sqrt{p}} \times \frac{c + d\sqrt{p}}{c + d\sqrt{p}}$$

Rule 3

If expression is in the following form or show some near resemblance to the following form

$$\frac{a + g\sqrt{b}}{c + d\sqrt{p}}$$

Find the conjugate value of the denominator whereby the sign is in between $c + d\sqrt{p}$ is flipped to negative and multiply conjugate value to both numerator and denominator to get the following result.

$$\frac{a + g\sqrt{b}}{c + d\sqrt{p}} \times \frac{c - d\sqrt{p}}{c - d\sqrt{p}}$$

As I have already implicitly showed the benefits of learning Indices in my Elementary Mathematics notes, which is that indices help us to reduce the amount of writing we require for certain expression which involves repeated multiplication or division of values.

The benefits of indices are quite blatantly obvious, but why should we learn surds as well then?

The definition of a surd is: a root (can be square root, cube root but we mostly focus on square roots in this stage) that cannot be written as a rational number. A famous example will be $\sqrt{2}$.

This is because the surd component you are going to read about serve as a basis for Additional Mathematics version of topics like trigonometry. There are certain commonly seen and frequently used values in trigonometry that output a surd form that will be very difficult to memorize if you write them in decimal format but memorizing them in surd form will be considerably easier, certain trigonometric equations is best to be manipulated while in surd form for easier, faster and more accurate calculations.

[Indices]

Question 1

1.1 Solve the following equations

(a) $5^{2x} - 24(5^x) - 25 = 0$

(b) $4(7^{2x}) = 25(7^x) + 21$

(c) $4(3^{2x+1}) + 11(3^x) - 5 = 0$

Solutions

(a) $5^{2x} - 24(5^x) - 25 = 0$

Substitute $u = 5^x$ (Remember to use the smallest coefficient of the exponent as substitution, example if given a choice between 5^{2x} and 5^x , we always select 5^x)

After the substitution, the equation can be rewritten in the following form

$$u^2 - 24u - 25 = 0$$

After using the quadratic formula, we get the following value for u

$$u = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(1)(-25)}}{2(1)}$$

$$u = -1 \text{ OR } u = 25$$

Since $u = 5^x$, we can rewrite in the following form:

$$5^x = -1 \text{ (Rejected) OR } 5^x = 25$$

Note: No positive or negative values of x within 5^x will produce a negative integer, thus $5^x = -1$ is rejected.

$$x = 2$$

(b) $4(7^{2x}) = 25(7^x) + 21$

Substitute $u = 7^x$ and rewrite in the following form.

$$4(u^2) = 25(u) + 21$$

$$4u^2 - 25u - 21 = 0$$

After using quadratic formula, we get the following value for u

$$u = \frac{-(-25) \pm \sqrt{(-25)^2 - 4(4)(-21)}}{2(4)}$$

$$u = 7 \text{ OR } u = -\frac{3}{4}$$

$7^x = 7$ OR $7^x = -\frac{3}{4}$ (Rejected, no positive or negative value of x within 7^x will produce a negative value)

$$x = 1$$

$$(c) 4(3^{2x+1}) + 11(3^x) - 5 = 0$$

The equation above can be rewritten as:

$$4(3^{2x})(3) + 11(3^x) - 5 = 0$$

$$12(3^{2x}) + 11(3^x) - 5 = 0$$

Substitute $u = 3^x$

$$12u^2 + 11u - 5 = 0$$

After using the quadratic formula, we get the following value for u

$$u = \frac{-(11) \pm \sqrt{11^2 - 4(12)(-5)}}{2(12)}$$

$$u = -\frac{5}{4} \text{ OR } u = \frac{1}{3}$$

$$3^x = \frac{1}{3} \text{ OR } 3^x = -\frac{5}{4} \text{ (Rejected)}$$

$$x = -1$$

1.2 Solve the following simultaneous equations.

$$2^{x+3y} = 8$$

$$9^{2x-y} = 27$$

Rewrite the equations as the following

$$2^{x+3y} = 2^3 \text{ [Equation 1]}$$

$$3^{4x-2y} = 3^3 \text{ [Equation 2]}$$

Extract out the powers we get the following

$$x + 3y = 3 \text{ [Equation 1A]}$$

$$4x - 2y = 3 \text{ [Equation 2A]}$$

$$x = 3 - 3y \text{ [Equation 1C]}$$

Substitute Equation 1C into Equation 1A

$$4(3 - 3y) - 2y = 3$$

$$12 - 12y - 2y = 3$$

$$12 - 14y = 3$$

$$-14y = 3 - 12 = -9$$

$$14y = 9$$

$$y = \frac{9}{14}$$

Substitute $y = \frac{9}{14}$ into Equation 1C

$$x = 3 - 3\left(\frac{9}{14}\right) = \frac{15}{14}$$

[Surds]

Question 2.

2.1 Simplify the following expression

$$(a) 11\sqrt{7} + 6\sqrt{28} - 5\sqrt{63}$$

$$(b) (4\sqrt{3} - \sqrt{2})(\sqrt{3} - 5\sqrt{2})$$

Solutions

$$(a) 11\sqrt{7} + 6\sqrt{28} - 5\sqrt{63}$$

$$\text{Rewrite as: } 11\sqrt{7} + 6\sqrt{7 \times 4} - 5\sqrt{9 \times 7}$$

$$\text{Once again, rewritten as } 11\sqrt{7} + 6(2)\sqrt{7} - 5(3)\sqrt{7}$$

Reason: Since $g\sqrt{ab^2} = gb\sqrt{a}$

Values like: $6\sqrt{28}$ can be expressed in the following form,

$$6\sqrt{7 \times 4} = (6 \times 2)\sqrt{7} = 12\sqrt{7}$$

We then proceed on to simplify the expression as

$$(11 + 12 - 15)\sqrt{7} = 8\sqrt{7}$$

$$(b) (4\sqrt{3} - \sqrt{2})(\sqrt{3} - 5\sqrt{2})$$

Expand the expression into the following

$$4\sqrt{3}(\sqrt{3}) - \sqrt{2}(\sqrt{3}) - 5\sqrt{2}(4\sqrt{3}) + 5\sqrt{2}(\sqrt{2})$$

$$4(3) - \sqrt{6} - 20\sqrt{6} + 5(2) = 12 + 10 - \sqrt{6} - 20\sqrt{6} = 22 - 21\sqrt{6}$$

2.2 Rationalize the denominator of the following

(a) $\frac{12}{\sqrt{3}}$

(b) $\frac{2-\sqrt{7}}{3+4\sqrt{7}}$

(c) $\frac{1}{3-\sqrt{5}}$

Solutions

(a) $\frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{(12\sqrt{3})}{3} = 4\sqrt{3}$

(b) $\frac{2-\sqrt{7}}{3+4\sqrt{7}} \times \frac{3-4\sqrt{7}}{3-4\sqrt{7}} = \frac{(2-\sqrt{7})(3-4\sqrt{7})}{(3+4\sqrt{7})(3-4\sqrt{7})}$

$$= \frac{[6 - 3\sqrt{7} - 8\sqrt{7} + 28]}{9 - 4^2(7)}$$

$$= \frac{6 + 28 - 3\sqrt{7} - 8\sqrt{7}}{-103} = \frac{34 - 11\sqrt{7}}{103} = \frac{-34}{103} - \frac{-(11)}{103}\sqrt{7}$$

$$= -\frac{34}{103} + \frac{11}{103}\sqrt{7}$$

(c) $\frac{1}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}$

$$= \frac{1(3-\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}$$

$$= \frac{3-\sqrt{5}}{9-5} = \frac{3}{4} + \frac{\sqrt{5}}{4}$$

2.3[Equations Involving Surds]

Step by Step Instructions for solving such equations

1. Rearrange the equation (if necessary) such that all square-roots are on the left-hand-side and all non-square-roots are on the right-hand-side.
2. Square both sides and then use the usual method you learn during Elementary Mathematics to solve equation (i.e. Rearrange and then use Quadratic Formula, etc.)
3. Substitute answers back into the original questions and reject values that don't make sense (i.e. square root of negative values, etc.)
4. Use the remaining values as answers. I typically leave the answers in surd form for Additional Mathematics paper unless the paper specifies something else.

$$(a) \sqrt{7x + 5} = x + 1$$

$$(b) 5\sqrt{5x + 9} - 4x - 3 = 0$$

$$(a) (\sqrt{7x + 5})^2 = (x + 1)^2$$

$$7x + 5 = x^2 + 2x + 1$$

$$0 = x^2 + 2x + 1 - 7x - 5$$

$$0 = x^2 - 5x - 4$$

After using the quadratic formula, we get the following

$$x = \frac{5}{2} - \frac{\sqrt{41}}{2} \text{ OR } x = \frac{5}{2} + \frac{\sqrt{41}}{2}$$

$$(b) 5\sqrt{5x + 9} - 4x - 3 = 0$$

Rearrange the equation such that square roots are on one side and non-square-roots are on the other side.

$$5\sqrt{5x + 9} = 4x + 3$$

$$(5\sqrt{5x + 9})^2 = (4x + 3)^2$$

$$25(5x + 9) = (4x)^2 + 2(3)(4x) + 9$$

$$125x + 225 = 16x^2 + 24x + 9$$

$$0 = 16x^2 + 24x - 125x + 9 - 225$$

$$0 = 16x^2 - 101x - 216$$

$$x = -\frac{27}{16} \text{ OR } x = 8$$

After substituting the value $x = -\frac{27}{16}$ into $5\sqrt{5x+9} - 4x - 3 = 0$

$$5\sqrt{5\left(-\frac{27}{16}\right) + 9} - 4\left(-\frac{27}{16}\right) - 3 = 7.5$$

Literally implies that $7.5 = 0$ (Reason for rejecting $x = -\frac{27}{16}$, substituting the value into the question yield illogical results.)

After substituting value $x = 8$ into $5\sqrt{5x+9} - 4x - 3 = 0$

$$5\sqrt{5(8) + 9} - 4(8) - 3 = 0$$

When we worked out the left side of the equation, we get $0 = 0$, which is within mathematical logic, therefore, we accept this as the only valid answer and thus $x = 8$

2.4 [Equality of Surds]

(a) Given that $a + b\sqrt{2} = (3 - \sqrt{2})^2 - \frac{8}{1 - \sqrt{2}}$

Find the value for a and b

$$(3 - \sqrt{2})^2 - \frac{8}{1 - \sqrt{2}} \times \frac{1 + \sqrt{2}}{1 + \sqrt{2}}$$

$$= 9 - 2(3)(\sqrt{2}) + 2 - \frac{[8(1 + \sqrt{2})]}{(1 - \sqrt{2})(1 + \sqrt{2})}$$

$$= 9 - 6\sqrt{2} + 2 - \frac{[8(1 + \sqrt{2})]}{(1 - \sqrt{2})(1 + \sqrt{2})}$$

$$= 11 - 6\sqrt{2} - \frac{8+8\sqrt{2}}{1^2-2}$$

$$= 11 - 6\sqrt{2} + 8 + 8\sqrt{2}$$

$$= 19 + 2\sqrt{2}$$

$$19 + 2\sqrt{2} = a + b\sqrt{2}$$

Therefore $b = 2$ and $a = 19$

Title	Additional Mathematics (Quadratic Equation)
Author(s)	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	6/3/2018

Mentoring Club is once again bringing free notes to the public. Before I start, I want to give those “first-timer” some hints and basic overview on what is covered in this topic. Unlike Elementary Mathematics, Additional Mathematics quadratic equation requires the following skills to be mastered as objective, aside from the usual quadratic formula and factorization you all do.

1. Sum and Product Manipulation
2. Discriminant Manipulation

Formula and Inequalities needed to remember as part of this topic:

Sum of Roots

$$\alpha + \beta = \frac{-b}{a}$$

Product of Roots

$$\alpha\beta = \frac{c}{a}$$

Proof of Concept Given Below

Since Quadratic Equation have Two Roots, we substitute the two roots as algebraic values.

$$\begin{aligned} x &= \alpha \quad \text{OR} \quad x = \beta \\ x - \alpha &= 0 \quad \text{OR} \quad x - \beta = 0 \\ (x - \alpha)(x - \beta) &= 0 \end{aligned}$$

When Expanded You Get

$$x^2 - \alpha x - \beta x + \alpha\beta = 0$$

And quadratic equations can be written in the following form

$$ax^2 + bx + c = 0$$

(In this case $a = 1$)

We extract the following values to derive the sum of roots

$$-\alpha x - \beta x = \frac{b}{a}x$$

Divide both sides by x to get

$$-\alpha - \beta = \frac{b}{a}$$

Dividing both sides by -1 to get

$$-1(-\alpha - \beta) = \frac{-b}{a}$$

Which can be simplified into

$$\alpha + \beta = \frac{-b}{a}$$

From the same equation, we can also derive the product of roots as well:

$$x^2 - \alpha x - \beta x + \alpha\beta = 0$$

In this case, we see how $\alpha\beta$ resembles c as

$$x^2 - \alpha x - \beta x + \alpha\beta = ax^2 + bx + c = 0$$

$$\text{Therefore } \alpha\beta = \frac{c}{a}$$

Principle of Discriminant:

$$D = b^2 - 4ac$$

Condition	Consequence and Interpretation
$D > 0$	Real and Distinct Roots
$D = 0$	Real and Equal Roots
$D \geq 0$	Real Roots (In General)
$D < 0$	Unreal Roots (Or called No Real Roots)

(Questions all Taken from Online Sources)

[Sum and Product of Roots]

(Warning: Write your notation clearly and in a way your teacher can at first glance tell what you are writing)

Q1. The equation $2x^2 - x - 2 = 0$ has roots α and β . **Without calculating the actual values of α and β .**

Find the values of

(a) $(\alpha + 3)(\beta + 3)$

(b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(c) $(\alpha - \beta)^2$

Steps Taken	Reasoning (If any)
Q1[General] $\alpha + \beta = \frac{1}{2}$ $\alpha\beta = -\frac{2}{2} = -1$	[Needed for the (a), (b), and (c)]
(a) $\alpha + 3$ $\beta + 3$ Since $(\alpha + 3)(\beta + 3) = \alpha\beta + 3\alpha + 3\beta + 9$ Which can be factorized into the following expression. $\alpha\beta + 3(\alpha + \beta) + 9$ Sub in values obtained previously and you should get $(-1) + 3\left(\frac{1}{2}\right) + 9 = \frac{18}{2}$	

(b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

Cross multiply the denominator to get

$$\frac{\alpha^2 + \beta^2}{\alpha\beta}$$

Which equates to the following values

$$\frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

Sub in previous values obtained and you should get

$$\frac{\left(\frac{1}{2}\right)^2 - 2(-1)}{(-1)}$$

$$= 2\frac{1}{4}$$

(c) $(\alpha - \beta)^2$

Expand to get the following

$$(\alpha^2 - 2\alpha\beta + \beta^2)$$

Rearrange the expression into the following pattern

$$\alpha^2 + \beta^2 - 2\alpha\beta$$

Which can be factored into

$$\alpha^2 + \beta^2 - 2(\alpha\beta)$$

From Question 1(b) we already found out the value

$$\text{for } (\alpha^2 + \beta^2) = -2\frac{1}{4}$$

Alongside with value of $\alpha\beta$ obtained earlier in the question.

Therefore

$$\begin{aligned}\alpha^2 + \beta^2 - 2\alpha\beta &= -2\frac{1}{4} - 2(-1) = -2\frac{1}{4} + 2 \\ &= -\frac{1}{4}\end{aligned}$$

Discriminant Manipulation

Question 2

Find the range of value for k for which the expression $3x^2 + 6x + k$ is **always positive** for all real value of x .

[When you see the following keywords: Always Positive, Always Negative, No Real Roots, it implies the graph does not have any contact with the X-axis, thus having no real roots and thus $b^2 - 4ac < 0$.]

Knowing such information, we will proceed on with finding a value of k that can satisfy the requirement of no real roots at all.

$$6^2 - 4(3)(k) < 0$$

$$36 - 12k < 0$$

$$-12k < -36$$

$$12k > 36$$

$$k > 3$$

Question 3

The expression $(k + 3)x^2 + 6x + k = 5$ has two distinct solutions for x .

(a) Show that k satisfy $k^2 - 2k - 24 < 0$

(b) Find the set of possible value of k

[When you see the following keywords, pass through graph at two distinct points, distinct roots or anything similar, it means the graph have two real and distinct roots, thus $b^2 - 4ac > 0$.]

(a)

$$6^2 - 4(k + 3)(k - 5) > 0$$

$$36 - 4(k^2 + 3k - 5k - 15) > 0$$

$$36 - 4k^2 - 12k + 20k + 60 > 0$$

$$-4k^2 - 12k + 20k + 60 - 36 > 0$$

$$-k^2 - 3k + 5k + 24 > 0$$

$$-k^2 + 2k + 24 > 0$$

$$k^2 - 2k - 24 < 0 \text{ [Shown]}$$

(b)

$$k^2 - 2k - 24 < 0$$

Factorize left hand side to get

$$(k - 6)(k + 4) < 0$$

(Less than zero means k must be within a range of values)

Thus, we get

$$k < 6 \text{ OR } k < -4$$

Which can be written as

$$-4 < k < 6$$

Question 3

Find the values of p for which the equation $3x^2 + 2px - p$ has

(a) Real and Equal Roots

(b) Distinct Roots

When you see the following keywords “has only 1 solution”, “real and equal roots”, “repeated roots” or anything similar the graph merely has contact with the X-axis one time, thus $b^2 - 4ac = 0$

(a)

$$(2p)^2 - 4(3)(-p) = 0$$

$$4p^2 + 12p = 0$$

$$p(p + 3) = 0$$

$$p = -3 \text{ or } p = 0$$

(b)

$$(2p)^2 - 4(3)(-p) > 0$$

$$4p^2 + 12p > 0$$

(More than zero means, value must not be within a range which means)

$$p > 0 \text{ OR } p < -3$$

Title	Exponents and Logarithms
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	23/9/2018

****This topic assumes you already understand your Elementary Mathematics Concepts.**

Applicable to the following courses/students.

- 'O' Level Additional Mathematics
- Nanyang Polytechnic – School of Chemical and Life Sciences – Mathematics for Life Sciences Module
- Nanyang Polytechnic – School of Engineering – Engineering Mathematics 1A Module
- Junior College/ A Levels – H1/H2 Mathematics
- Institute of Technical Education (ITE Colleges) – Technical Mathematics Module (Specific Courses and Levels Only)

Relationship Between Exponents and Logarithm as Follows	
An expression written in Exponential Form	$3^{12} = 531441$
If written as Logarithmic Format, it looks like this	$\log_3 531441 = 12$

Generally Speaking	
If	$a^b = c$ In this case, b is called the exponent or power a is called the base value.
It can be rewritten as	$\log_a c = b$ In this case b is known as the logarithm a is called the base value

Laws of Logarithm	
$\log_a(xy) = \log_a x + \log_a y$	
$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	
$\log_a(x)^y = y \log_a(x)$	

$\log_a x = \frac{\log_b(x)}{\log_b(a)}$	Calculator Input [Calculator assumes the input is in base value is 10 by default] If $a^x = y$ You literally type in the following into the calculator to find x $\frac{\log(y)}{\log(a)} = x$
--	---

Relationship Between Logarithm and Exponents as follows

If $a^x = y$

$$x = \log_a(y)$$

If $\log_a x = y$

$$x = a^y$$

If you are thinking about calculator input, the following property will be useful if needed. (As said before, calculator assumes the $\log(x)$, as logarithm of which base is 10.)

$$a^x = y$$

$$x = \frac{\log(y)}{\log(a)}$$

$$\log(x) = y$$

$$x = 10^{\log(x)} = 10^y$$

Motivations for introducing logarithm and modern use of logarithm includes the following

- Historical use of logarithm includes multiplication of large numbers, which can be converted to adding of their logarithm under a certain base value and getting the final value
- Modern use of logarithm includes, astronomy (studying brightness of stars), computation (performing data encryption) and more.

Introducing concept of base e , also known as Euler's Number.

- e is approximately 2.71828 when rounded of to 6 significant figures.
- e can be approximated using the formula as shown below

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \dots \dots \frac{1}{n!}$$

$$n! = n(n-1)(n-2)(n-3) \dots \dots (3)(2)(1)$$

- The approximation gets better as more terms are added.
- Used in modelling continuous growth or continuous decay in Mathematics and Sciences.
- Modern scientific calculators have this functionality, just locate button e^x or e on your scientific calculator.

Introducing Natural Logarithm ($\ln(x)$)

- Natural Logarithm assumes that the base of a certain exponent is exactly e
- It finds the logarithm where the value has base value is e

The relationship between $\ln(x)$ and e^x goes as follows

If, $e^x = y$ and you are being asked to find the value of x

$$\ln(e^x) = \ln(y)$$

$$x = \ln(y)$$

If, $\ln(x) = y$ and you are being asked to find the value of x

$$e^{\ln(x)} = e^y$$

$$x = e^y$$

(For students studying higher level Mathematics, your lecturers may tell you Logarithm an inverse function of Exponent and vice-versa is true as well, this is because, inverse function sends a function back to where it originally started.)

Solving equations involving Logarithm and Exponents

(The following questions are provided by my friend in Nanyang Polytechnic who is studying Electrical Engineering related courses, he explained they are taken from his Engineering Mathematics Notes.)

If questions specify $\ln(x)$ or something similar, the base is exactly e . If question specify $\log(x)$ without base value information, the base value is to be assumed to be 10.

Example 1.

(a) $2^x = 16$

(b) $5^x = 0.3$

(c) $e^{-x} = 17.54$

$$(d) \log(x) = 2.3$$

$$(e) \log(2x) + \log(x) = 9$$

Solutions

1(a)

$$2^x = 16$$

$$\log_2 16 = x = 4$$

1(b)

$$5^x = 0.3$$

$$\log_{0.3} 5 = -0.748$$

1(c)

$$e^{-x} = 17.54$$

$$\ln(e^{-x}) = \ln(17.54)$$

$$-x = \ln(17.54) = 2.86$$

$$x = -2.86$$

1(d)

$$\log(x) = 2.3$$

$$10^{\log(x)} = 10^{2.3}$$

$$x = 10^{2.3} = 200$$

1(e)

$$\log(2x) + \log(x) = 9$$

Applying law of logarithm, we get

$$\log(2x) + \log(x) = \log(2x^2) = 9$$

$$10^{\log(2x^2)} = 2x^2 = 10^9$$

$$2x^2 = 10^9$$

$$x^2 = \frac{10^9}{2}$$

$$x = \sqrt{\frac{10^9}{2}}$$

Example 2

(a) $2^{2x} - 8(2^x) + 15 = 0$

(b) $e^{2x} - 7e^x + 12 = 0$

(c) $e^{e^x} = 3$

Depending on the situation, you sometimes may have to reject values, you should always substitute back into the original equation or check with the original question if the values make sense in the question's context.

[I always use u to perform substitution, you can use any letter you like but make sure you don't get yourself confused afterwards.]

Solutions

2(a)

$$2^{2x} - 8(2^x) + 15 = 0$$

Substitute $u^x = 2^x$ (In this case, I pick the smallest power component within the equation to substitute.)

The equation effectively becomes the following after I substituted the component values.

$$u^2 - 8u + 15 = 0$$

Using Quadratic Formula, which I won't elaborate further on.

I get the following roots for u

$$u = 3 \text{ OR } u = 5$$

The roots can be written as the following values since $u = 2^x$ as defined earlier

$$2^x = 3 \text{ OR } 2^x = 5$$

$$x = \log_2(3) \text{ OR } x = \log_2(5)$$

$$x = 1.584962 \text{ OR } x = 2.321928$$

$$2(b) e^{2x} - 7e^x + 12 = 0$$

Substitute $u = e^x$

Equation becomes the following

$$u^2 - 7u + 12 = 0$$

Using quadratic formula, we can get the following answers

$$u = 3 \text{ OR } u = 4$$

Which can be rewritten as

$$e^x = 3 \text{ OR } e^x = 4$$

Taking natural logarithm on both sides of each solution to arrive at value of x

$$\ln(e^x) = \ln(3) \text{ OR } \ln(e^x) = \ln(4)$$

$$x = \ln(3) = 1.098612 \text{ OR } x = \ln(4) = 1.386294$$

$$2(c) e^{e^x} = 3$$

Take natural logarithm on both sides,

$$\ln(e^{e^x}) = \ln(3)$$

We get the following equation

$$e^x = \ln(3)$$

Take natural logarithm on both sides once again the get the following,

$$\ln(e^x) = \ln(\ln(3))$$

$$x = \ln(\ln(3))$$

$$x = 0.094048$$

Title	Binomial Expansion and Binomial Theorem
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	1/6/2018

Our learning objective

- To learn how to read the notation of binomial expansion
- To learn how to use binomial theorem for various calculations

Application of Binomial Expansion includes the following:

Serve as a good foundation for studying

- H2 Mathematics – Statistics
- Engineering – Engineering Mathematics (Statistics)
- Mathematics for Life Sciences – Biostatistics
- Information Technology – Computing Mathematics (Statistics)
- Business – Business Statistics

As Additional Mathematics itself isn't a compulsory subject or prerequisite for Polytechnic courses, those going for polytechnic can still study statistics module at higher level and the below will be mentioned in your lecture. However, those going for Junior College, it is important to take this seriously as the Mathematics in H2 is considerably harder than the Polytechnic's Statistical Analysis module, without such foundation, catching up is rather difficult if not impossible.

While the notation for binomial expansion varies in different textbooks, they literally, mean the same thing. I will try my best to explain the various ways binomial expansion formula is being written and explain how they correlate the each other.

Standard Universal Binomial Expansion Notation

$(a + b)^n$	Multiply $(a + b)$ by itself n number of times
$\binom{n}{r}$	From n number of objects, you choose r of them. The value $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, is the number of combinations or the number of ways you can arrange r objects, assuming the <u>order doesn't matter</u>. To calculate $\binom{n}{r}$ on your scientific calculator, you press the value of n, followed by "C", followed by value of r (Ask teachers if in doubt) *** Various online resources may write this notation in a different way. Notations like $n C r$ mean the same thing, it meant from n number of objects, choose r of them and calculate the number of combinations.
$n!$	Known as n factorial. Multiply the integer value of n, $(n - 1)$, $(n - 2)$ all the way to (3), (2), (1) Note: n must be an integer and $n \geq 0$ ($n < 0$ is undefined) Example: $5! = 5 \times 4 \times 3 \times 2 \times 1$ If you get an error message while computing factorials in your calculator, check the following and correct them accordingly. (Casio FX-95 SG Plus and Casio FX-96 GS PLUS Calculator) Syntax Error ➤ Check if you keyed in something wrongly into the calculator, anything from, missing values, forget to close bracket, etc. Math Error ➤ Check if you accidentally keyed in negative numbers, followed by the factorial button. (Factorials of negative numbers are undefined) ➤ Check if you accidentally keyed in a value with decimal places, followed by the factorial button (Factorials of decimal digits are invalid) ➤ Check if you accidentally keyed in a value ≥ 70, followed by the factorial button (Calculators are unable to compute Factorials of 70 or greater)

Local Standard Notation

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 \dots \binom{n}{r} a^{n-r} b^r \dots + b^n$$

Expression to be
evaluated

1st term

2nd term

3rd term

$(r + 1)th$ term

(n) th term

Binomial Expansion General Term Formula as Follows
$T_{r+1} = \binom{n}{r} a^{n-r} b^r$

Examples as Follows

Example 1:

Given the following binomial expansion, write down the first 3 terms of the expansion, in the simplest form possible.

$$(x + 5)^{12}$$

Approaching the question using General Term Formula:

$$T_1 = \binom{12}{0} x^{12} (5^0) = x^{12}$$

$$T_2 = \binom{12}{1} x^{12-1} (5^1) = 60x^{11}$$

$$T_3 = \binom{12}{2} x^{12-2} (5^2) = 66(25)x^{10} = 1650x^{10}$$

First three terms as written as below is:

$$x^{12}, 60x^{11}, 1650x^{10}$$

Title	Additional Mathematics ['N' Levels] and Computing Mathematics Calculus - Basic Differentiation
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	5/4/2018

This document is suitable for students of the following categories

- Nanyang Polytechnic School of Information Technology Students [G1]
- 'N' Levels Additional Mathematics Students [G2]

This document is however not useful for students studying in the following fields for the following reasons

- 'O' Level and 'A' Level (Trigonometric, Exponential and Logarithmic Function are not covered in this document)
- Engineering (Trigonometric Function Not Covered)
- Chemistry (Logarithmic and Exponential Function Not Covered)

However, if you haven't studied additional mathematics before and want exposure to the topic before entering your course or before you enter Secondary 4, this should give you the basic concepts of how it works.

The following are covered in this document

Basic Overview of Differentiation

- Reason to study this topic
- Notation Guide

Differentiation Basic Rules

- Power Rule
- Sum Rule
- Difference Rule
- Product Rule
- Quotient Rule
- Chain Rule

Brief Overview of Differentiation (And Calculus in General)

Without going into too much detail about the history of calculus, calculus is a branch of Mathematics introduced to study about change. Differentiation is a sub-branch that deals with the gradient of a given function at every point on a curve.

Calculus is very important in the following field for the following reasons

- Computing – To study time complexity of programming algorithms and do various data analytics and estimation
- Physics – Majority of physical laws are derived with help of calculus (Newton used calculus to research how physics works and used that in his later works)
- Engineering – Engineering is closely related to physics as Engineering students study physics as their basic module before they carry on with intermediate modules
- Chemistry – Study speed of chemical reactions

And the list can go on and on

Notation	Meaning	Explanation	Notes
$F(x)$	Function of x		I also use $f(x), g(x)$ in certain situations
$F'(x)$	First Derivative of $f(x)$		I also use $f'(x), g'(x)$ in certain situations
$F''(x)$	Second Derivative of $f(x)$	First derivative of $f'(x)$	I also use $f''(x), g''(x)$ in certain situations
$\frac{dy}{dx}$	First derivative of y with respect to x		
$\frac{d^2y}{dx^2}$	Second derivative of y with respect to x	First derivative of $\frac{dy}{dx}$	

Follow the question with regards to notation matters:

If the question mentions a function in the following form:

$$f(x) = ax^n + ax^{n-1} \dots + c$$

When answering the question, use $f'(x)$ for first derivative and $f''(x)$ for second derivative.

If the question mentions an equation in the following form:

$$y = ax^n + ax^{n-1} \dots + c$$

When answering the question use $\frac{dy}{dx}$ for first derivative and $\frac{d^2y}{dx^2}$ for second derivative

Power Rule

Given Function $f(x) = ax^n$

The derivative of the function is $f'(x) = anx^{n-1}$

Question 1.

Differentiate the following with respect to x

(a) $y = x^5$

(b) $y = \frac{1}{x^3}$

(c) $y = 3x^4$

(d) $y = \pi$

1(a)

$$y = x^5$$

Differentiate using power rule $\frac{dy}{dx} x^5 = 5x^4$

1(b)

$$y = \frac{1}{x^3}$$

Convert to index notation: $y = x^{-3}$

Differentiate using power rule: $\frac{dy}{dx} x^{-3} = -3x^{-4}$

1(c)

$$y = 3x^4$$

Differentiate using power rule: $\frac{dy}{dx} 3x^4 = 12x^3$

1(d)

$$y = \pi$$

Derivative of any constant value is 0: $\frac{dy}{dx} \pi = 0$

Sum Rule

Given Function $F(x) = f(x) + g(x)$

The derivative is $F'(x) = f'(x) + g'(x)$

I know in notation form it is rather complicated so I am going to explain it in relatively simple terms.

Imagine you have the following function

$$f(x) = ax^n + bx^m$$

To find the derivative, you must first split the function into two sections:

First Section	Second Section
ax^n	bx^m

Differentiate both separately to get

Derivative of first section	Derivative of second section
anx^{n-1}	$bm x^{m-1}$

Combine them back together to get your “overall” derivative

$f'(x) = anx^{n-1} + bm x^{m-1}$

If you still don't get it, take note of the following examples, often seeing how a real question work out clears most of your doubts.

Question 2

Differentiate the following with respect to x

(a) $y = 6x^7 + 2x^3$

(b) $y = 15x^2 + 4x^{-2} + \frac{1}{x}$

2(a)

Split the equation into two sections:

$$\frac{dy}{dx} 6x^7 + 2x^3 = \frac{d}{dx} 6x^7 + \frac{d}{dx} 2x^3$$

Differentiate the components one by one to get

$\frac{d}{dx} 6x^7 = 42x^6$	$\frac{d}{dx} 2x^3 = 6x^2$
-----------------------------	----------------------------

Combine them back together to get the following

$$\frac{dy}{dx} = 42x^6 + 6x^2$$

2(b)

Convert all to index notation

$$15x^2 + 4x^{-2} + x^{-1}$$

Split equation into three sections

$$\frac{dy}{dx} 15x^2 + 4x^{-2} + x^{-1} = \frac{d}{dx} 15x^2 + \frac{d}{dx} 4x^{-2} + \frac{d}{dx} x^{-1}$$

Differentiate the components one by one to get

$\frac{d}{dx} 15x^2 = 30x$	$\frac{d}{dx} 4x^{-2} = -8x^{-3}$	$\frac{d}{dx} x^{-1} = -1(x)^{-2} = -x^{-2}$
----------------------------	-----------------------------------	--

Combine them to get

$$\frac{dy}{dx} = 30x - 8x^{-3} - x^{-2}$$

Difference Rule

Given Function $F(x) = f(x) - g(x)$

The derivative is $F'(x) = f'(x) - g'(x)$

Imagine the following function:

$$f(x) = ax^n - bx^m$$

To find the derivative split the function into 2 sections

First Section	Second Section
ax^n	$-bx^m$

Differentiate both section separately to get

anx^{n-1}	$-bmx^{m-1}$
-------------	--------------

Combine them back together to get your "overall" derivative

$f'(x) = anx^{n-1} - bmx^{m-1}$

Question 3:

Differentiate the following with respect to x

(a) $y = 5x^7 - 2x^3 - 7$

(b) $y = 3x^4 - 4x^2 + 5x^{-4}$

3(a)

Split equation into 3 sections to get

$$\frac{dy}{dx} 5x^7 - 2x^3 - 7 = \frac{d}{dx} 5x^7 - \frac{d}{dx} 2x^3 - \frac{d}{dx} 7$$

Differentiate the components one by one to get

$\frac{d}{dx} 5x^7 = 35x^6$	$\frac{d}{dx} - 2x^3 = -6x^2$	$\frac{d}{dx} 7 = 0$
-----------------------------	-------------------------------	----------------------

Combine them back and you should get the following

$$\frac{dy}{dx} = 35x^6 - 6x^2$$

3(b)

Split equation into 3 sections to get

$$\frac{dy}{dx} 3x^4 - 4x^2 + 5x^{-4} = \frac{d}{dx} 3x^4 - \frac{d}{dx} 4x^2 + \frac{d}{dx} 5x^{-4}$$

Differentiate components one by one to get

$\frac{d}{dx} 3x^4 = 12x^3$	$\frac{d}{dx} -4x^2 = -8x$	$\frac{d}{dx} 5x^{-4} = -20x^{-5}$
-----------------------------	----------------------------	------------------------------------

Combine them back to get the following:

$$\frac{dy}{dx} = 12x^3 - 8x - 20x^{-5}$$

Product Rule

Given Function $F(x) = f(x) \cdot g(x)$

The derivative is $F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Imagine the following function

$$f(x) = (ax^n)(bx^m)$$

$$f'(x) = anx^{n-1}(bx^m) + ax^n(bmx^{m-1})$$

Question 4

Differentiate the following with respect to x

(a) $y = (x^2 + 15)(6x^4 + 9)$

(b) $y = (5x^9 + 6x - 5)(2x^2 + 6)$

4(a)

$$y = (x^2 + 15)(6x^4 + 9)$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 15) \cdot (6x^4 + 9) + \frac{d}{dx}(6x^4 + 9) \cdot (x^2 + 15)$$

$$\begin{aligned}\frac{dy}{dx} &= 2x(6x^4 + 9) + 24x^3(x^2 + 15) \\ &= 12x^5 + 18x + 24x^5 + 360x^3 \\ &= 36x^5 + 360x^3 + 18x\end{aligned}$$

4(b)

$$y = (5x^9 + 6x - 5)(2x^2 + 6)$$

$$\frac{dy}{dx} = \frac{d}{dx}(5x^9 + 6x - 5) \cdot (2x^2 + 6) + \frac{d}{dx}(2x^2 + 6) \cdot (5x^9 + 6x - 5)$$

$$\frac{dy}{dx} = (45x^8 + 6)(2x^2 + 6) + (4x)(5x^9 + 6x - 5)$$

$$\begin{aligned}&= (90x^{10} + 12x^2 + 270x^8 + 36) + (20x^{10} + 24x^2 - 20x) \\ &= 110x^{10} + 270x^8 + 36x^2 - 20x + 36\end{aligned}$$

Quotient Rule

Given function $F(x) = \frac{f(x)}{g(x)}$

The derivative is $F'(x) = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{[g(x)]^2}$

Imagine you have the following function

$$f(x) = \frac{ax^n}{bx^n}$$

$$f'(x) = \frac{anx^{n-1}(bx^n) - bnx^{n-1}(ax^n)}{(bx^n)^2}$$

Question 5

Differentiate the following with respect to x

$$(a) y = \frac{2x^5 + 3x^2 - 7}{(2x + 6)}$$

$$y = \frac{2x^5 + 3x^2 - 7}{(2x + 6)}$$

$$\frac{dy}{dx} = \frac{\left[\frac{d}{dx}(2x^5 + 3x^2 - 7) \cdot (2x + 6) - \frac{d}{dx}(2x + 6) \cdot (2x^5 + 3x^2 - 7) \right]}{(2x + 6)^2}$$

$$\frac{dy}{dx} = \frac{[(10x^4 + 6x)(2x + 6) - 2(2x^5 + 3x^2 - 7)]}{(2x + 6)^2}$$

$$\frac{dy}{dx} = \frac{20x^5 + 12x^2 + 60x^4 + 36x - 4x^5 - 6x^2 + 14}{(2x + 6)^2}$$

$$\frac{dy}{dx} = \frac{16x^5 + 6x^2 + 60x^4 + 36x + 14}{(2x + 6)^2}$$

$$\frac{dy}{dx} = \frac{16x^5 + 60x^4 + 6x^2 + 36x + 14}{(2(x + 3))^2}$$

$$\frac{dy}{dx} = \frac{16x^5 + 60x^4 + 6x^2 + 36x + 14}{4(x + 3)^2}$$

$$\frac{dy}{dx} = \frac{8x^5 + 30x^4 + 3x^2 + 18x + 7}{2(x + 3)^2}$$

Chain Rule

Given function

$$F(x) = [f(x) + g(x)]^c \quad \{\text{where } c \text{ is a constant.}\}$$

The derivative is

$$F'(x) = c[f(x) + g(x)]^{c-1} \cdot [f'(x) + g'(x)]$$

Imagine you have the following function

$$f(x) = (ax^n + bx^m)^c$$

The derivative is

$$f'(x) = c(ax^n + bx^m)^{c-1} [anx^{n-1} + bmx^{m-1}]$$

Question 6

Differentiate the following with respect to x

$$(a) \ y = (2x^3 - 5x^2 + 2x)^7$$

$$y = (2x^3 - 5x^2 + 2x)^7$$

$$\frac{dy}{dx} = 7(2x^3 - 5x^2 + 2x)^{7-1} \cdot (6x^2 - 10x + 2)$$

$$\frac{dy}{dx} = 7(2x^3 - 5x^2 + 2x)^6 (6x^2 - 10x + 2)$$

Tips and Tricks Below

Question 7

Differentiate the following with respect to x

$$y = \frac{1}{(x+9)^5}$$

Question: Should I use chain rule or quotient rule for this question?

Answer: Chain rule, it is easier to use chain rule as demonstrated below.

Reason Below:

By Chain Rule (3 Steps)

$$y = \frac{1}{(x+9)^5} = (x+9)^{-5}$$

$$\frac{dy}{dx} = -5(x+9)^{-6}(1)$$

$$\frac{dy}{dx} = -5(x+9)^{-6}$$

By Quotient Rule (4 Steps):

$$y = \frac{1}{(x+9)^5}$$

$$\frac{dy}{dx} = \frac{[0(x+9)^5 - 5(x+9)^{5-1}(1)]}{(x+9)^{5(2)}}$$

$$\frac{dy}{dx} = \frac{-5(x+9)^4}{(x+9)^{10}}$$

$$\frac{dy}{dx} = -5(x+9)^{4-10}$$

$$\frac{dy}{dx} = -5(x+9)^{-6}$$

Conclusion: If the numerator = 1 and the denominator is in the form of $(ax^n + bx^m \dots + c)$, you just have to convert the expression or equation into index notation and use chain rule to differentiate. (This only applies when the numerator = 1.)

Title	Differentiation of Exponential and Logarithmic Function
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	20/12/2018

This article assumes you know and understand the notation for differentiation.

Applicable to

- Singapore-Cambridge Secondary School GCE 'O' Level Additional Mathematics
- Nanyang Polytechnic, School of Chemical and Life Sciences – Mathematics for Life Science Module
- Nanyang Polytechnic, School of Engineering – Engineering Mathematics 1A Module
- Singapore-Cambridge Junior College GCE 'A' Level H1 Mathematics
- Singapore-Cambridge Junior College GCE 'A' Level H2 Mathematics (Bridging Only)

Chain Rule (In General)
Given $f(x) = g(h(x))$
The derivative is given as $f'(x) = [g'(x)][h'(x)]$

General Cases for Derivatives of Exponential and Logarithmic Functions [Derived by Applying Chain Rule to the Specific Cases Mentioned Further Below.]	
Function	Derivatives
$f(x) = a^{g(x)}$	$f'(x) = [a^{g(x)} \ln(a)]g'(x)$
$f(x) = ce^{g(x)}$ Where c is a coefficient of $e^{g(x)}$, with e representing Euler's Number.	$f'(x) = [ce^{g(x)}]g'(x)$
$f(x) = \ln(g(x))$	$f'(x) = \frac{1}{g(x)} [g'(x)] = \frac{g'(x)}{g(x)}$

Specific Case	Derivatives
$f(x) = e^x$	$f'(x) = e^x$
$f(x) = ce^x$	$f'(x) = ce^x$
$f(x) = a^x$	$f'(x) = a^x \ln(a)$
$f(x) = \ln(x)$	$f'(x) = \frac{1}{x}$

Question 1.

Differentiate the following with respect to x

- (a) e^{6x}
- (b) $7e^{2x+6}$
- (c) 5^{7x}
- (d) 3^{2x+6}
- (e) $\ln(6x^2 + 5x)$

Question 1

(a) $\frac{d}{dx} e^{6x}$

Rewrite as

$f(x) = e^{6x}$

$g(x) = e^{6x}$	$g'(x) = e^{6x}$
$h(x) = 6x$	$h'(x) = 6$
$f'(x) = 6e^{6x}$	
$\frac{d}{dx} = 6e^{6x}$	

(b) $\frac{d}{dx} 7e^{2x+6}$

Rewrite as

$f(x) = 7e^{2x+6}$

$g(x) = 7e^{2x+6}$	$g'(x) = 7e^{2x+6}$
$h(x) = 2x + 6$	$h'(x) = 2$
$f'(x) = 14e^{2x+6}$	
$\frac{d}{dx} = 14e^{2x+6}$	

(c) $\frac{d}{dx} 5^{7x}$

Rewrite as
 $f(x) = 5^{7x}$

$g(x) = 5^{7x}$	$g'(x) = 5^{7x} \ln(5)$
$h(x) = 7x$	$h'(x) = 7$
$f'(x) = 7 (5^{7x}) \ln(5)$	
$\frac{d}{dx} = 7(5^{7x}) \ln(5)$	

(d)
 $\frac{d}{dx} 3^{2x+6}$

Rewritten as $f(x) = 3^{2x+6}$	
$g(x) = 3^{2x+6}$	$g'(x) = 3^{2x+6} \ln(3)$
$h(x) = 2x + 6$	$h'(x) = 2$

$\frac{d}{dx} = 2(3^{2x+6} \ln(3))$

(e)

$$\frac{d}{dx} \ln(6x^2 + 5x)$$

For students who can remember the formula for differentiation of $\ln[g(x)]$, go ahead and use the formula, unfortunately, this might not be the case for all students and I still want to demonstrate the more “tedious” approach. However, please note that the more tedious method requires less memory work.

Rewritten as:

$$f(x) = \ln(6x^2 + 5x)$$

$$f'(x) = \frac{1}{6x^2+5x} (12x + 5)$$

$$f'(x) = \frac{(12x+5)}{6x^2+5x} \quad \frac{d}{dx} = \frac{12x+5}{6x^2+5x}$$

Title	Additional Mathematics ['N' Levels] and Computing Mathematics Calculus – Basic Integration and Indefinite Integrals
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	7/4/2018

This document is suitable for students of the following categories

- Nanyang Polytechnic School of Information Technology Students [G1]
- 'N' Levels Additional Mathematics Students [G2]

This document is however not useful for students studying in the following fields for the following reasons

- 'O' Level and 'A' Level (Trigonometric, Exponential and Logarithmic Function are not covered in this document)
- Engineering (Trigonometric Function Not Covered)
- Chemistry (Logarithmic and Exponential Function Not Covered)

However, if you haven't studied additional mathematics before and want exposure to the topic before entering your course or before you enter Secondary 4, this should give you the basic concepts of how it works.

The following are covered in this release:

Basic Overview of Integration

- Integration as the reverse of differentiation
- Notation guide
- Indefinite integral

Rules of integration

- Constant Multiple Rule
- Sum Rule
- Difference Rule

***Remember to print page 3 if you need something to refer to while doing your assignments. This material is not designed in the same way as the differentiation topic notes I created previously.

Integration as the reverse process of differentiation

As covered in earlier guide, we discussed how differentiation works, the reason why we study integration is the following reasons:

- There will be situations where finding the “anti-derivative” (AKA integral) is useful, if you are only given the gradient and ask to find out what is the equation of the function.
- Further study of integration will also be applied to finding the area under the graph of the function, using definite integral, which is useful for solving certain problems in various fields like Physics, Engineering, Chemistry and Information Technology.

Notation of Indefinite Integral

The diagram shows the equation $\int x^n dx = \frac{x^{n+1}}{n+1} + C$. A blue arrow points from the integral symbol $\int$ to a box labeled “Indefinite Integral of Function” Symbol. Another blue arrow points from the constant C to a box labeled Arbitrary Constant.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

“Indefinite Integral of Function” Symbol

Arbitrary Constant

Explanation

Arbitrary Constant: The reason why we must have a “+C” to any indefinite integral is

- Reversing the integration process by differentiating the integral will result in many possible values of the constants that satisfy the integral, thus a “+C” is added, to represent “Constant”

Rules of Integration:

Integration of Constant Values

$$\int k \, dx = kx + C$$

Constant Multiple (Power Function Integration) Rule (Provided $n \neq -1$):

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$\int kx^n \, dx = k \int x^n = k \left(\frac{x^{n+1}}{n+1} \right) + C = \frac{kx^{n+1}}{n+1} + C$$

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \quad (\text{Provided } a \neq 0 \text{ AND } n \neq -1)$$

Generalized Sum Rule

$$\int k \cdot f(x) + h \cdot g(x) = k \int f(x) + h \int g(x) + C$$

Generalized Difference Rule

$$\int k \cdot f(x) - h \cdot g(x) = k \int f(x) - h \int g(x) + C$$

Generalization of Constant Multiple, Sum and Difference Rule (Provided $n \neq -1$):

$$\int kx^n \pm gx^m = k \left(\frac{x^{n+1}}{n+1} \right) \pm g \left(\frac{x^{m+1}}{m+1} \right) + C = \frac{kx^{n+1}}{n+1} \pm \frac{gx^{m+1}}{m+1} + C$$

Indefinite Integral

Example 1:

Find the following integrals [Power Function Integration Rule]

(a) $\int x^6 dx$

(b) $\int \frac{1}{x^8} dx$

$$1(a) \int x^6 dx = \frac{x^{6+1}}{6+1} = \frac{x^7}{7} + C$$

$$1(b) \int \frac{1}{x^8} dx = \int x^{-8} = \frac{x^{-8+1}}{-8+1} = \frac{x^{-7}}{-7} + C = -\frac{1}{7}x^{-7} + C$$

Example 2:

Find the following integrals [Integration of Constant Values]

(a) $\int 15 dx$

(b) $\int 25 dx$

$$2(a) \int 15 dx = 15x + C$$

$$2(b) \int 25 dx = 25x + C$$

Example 3:

Find the following integrals [Constant-Multiple (Power Function Integration) Rule]

(a) $\int 2x^5 dx$

(b) $\int 16x^7 dx$

$$3(a) \int 2x^5 dx$$

$$\int 2x^5 dx = 2 \int x^5 dx + C = 2 \left(\frac{x^{5+1}}{5+1} \right) + C = \frac{2x^{5+1}}{6} = \frac{2x^6}{6} + C = \frac{1}{3}x^6 + C$$

$$3(b) \int 16x^7 dx$$

$$\int 16x^7 dx = 16 \int x^7 dx = 16 \left(\frac{x^{7+1}}{7+1} \right) + C = \frac{16x^8}{8} + C = 2x^8 + C$$

Example 4:

Find the following integrals [\[Generalized Sum Rule\]](#)

(a) $\int 2x^9 + 5x^3 dx$

(b) $\int 9x^7 + 6x^{-8} + 12x + 7 dx$

4(a) $\int 2x^9 + 5x^3 dx$

$$= 2 \int x^9 + 5 \int x^3$$

$$= 2 \left(\frac{x^{9+1}}{9+1} \right) + 5 \left(\frac{x^{3+1}}{3+1} \right) + C$$

$$= 2 \left(\frac{x^{10}}{10} \right) + 5 \left(\frac{x^4}{4} \right) + C$$

$$= \frac{1}{5} x^{10} + \frac{5}{4} x^4 + C$$

4(b) $\int 9x^7 + 6x^{-8} + 12x + 7 dx$

$$= 9 \int x^7 + 6 \int x^{-8} + 12 \int x + \int 7$$

$$= 9 \left(\frac{x^{7+1}}{7+1} \right) + 6 \left[\frac{x^{(-8)+1}}{(-8)+1} \right] + 12 \left(\frac{x^{1+1}}{1+1} \right) + 7x + C$$

$$= \frac{9}{8} x^8 + 6 \left(\frac{x^{-7}}{-7} \right) + \frac{12x^2}{2} + 7x + C$$

$$= \frac{9}{8} x^8 - \frac{6}{7} x^{-7} + 6x^2 + 7x + C$$

Example 5

Find the following integrals [\[Generalized Difference Rule\]](#)

$$(a) \int 2x^2 - 9x - \frac{2}{x^9} - 8 \, dx$$

5(a)

$$\int 2x^2 - 9x - \frac{2}{x^9} - 8 \, dx$$

$$= 2 \int x^2 - 9 \int x - 2 \int x^{-9} - \int 8$$

$$= 2 \left(\frac{x^{2+1}}{2+1} \right) - 9 \left(\frac{x^{1+1}}{1+1} \right) - 2 \left(\frac{x^{-9+1}}{-9+1} \right) - 8(x) + C$$

$$= 2 \left(\frac{x^3}{3} \right) - 9 \left(\frac{x^2}{2} \right) - 2 \left(\frac{x^{-8}}{-8} \right) - 8(x) + C$$

$$= 2 \left(\frac{x^3}{3} \right) - 9 \left(\frac{x^2}{2} \right) + 2 \left(\frac{x^{-8}}{8} \right) - 8x + c$$

$$= \frac{2}{3}x^3 - \frac{9}{2}x^2 + \frac{2}{8}x^{-8} - 8x + C$$

$$\frac{2}{3}x^3 - \frac{9}{2}x^2 + \frac{1}{4}x^{-8} - 8x + C$$

Example 6

Find the following integrals [\[Generalization of all Rules Mentioned Combined\]](#)

$$(a) \int 3x^5 - 6x^4 + 11x - 16 + 9x^{-7} dx$$

$$3 \int x^5 - 6 \int x^4 + 11 \int x - \int 16 + 9 \int x^{-7}$$

$$3 \left(\frac{x^{5+1}}{5+1} \right) - 6 \left(\frac{x^{4+1}}{4+1} \right) + 11 \left(\frac{x^{1+1}}{1+1} \right) - 16(x) + 9 \left(\frac{x^{-7+1}}{-7+1} \right) + C =$$

$$3 \left(\frac{x^6}{6} \right) - 6 \left(\frac{x^5}{5} \right) + 11 \left(\frac{x^2}{2} \right) - 16x + 9 \left(\frac{x^{-7+1}}{-7+1} \right) + C =$$

$$\frac{3}{6}x^6 - \frac{6}{5}x^5 + \frac{11}{2}x^2 - 16x + \frac{9x^{-6}}{(-6)} + C =$$

$$\frac{1}{2}x^6 - \frac{6}{5}x^5 + \frac{11}{2}x^2 - 16x - \frac{9}{6}x^{-6} + C =$$

$$\frac{1}{2}x^6 - \frac{6}{5}x^5 + \frac{11}{2}x^2 - 16x - \frac{3}{2}x^{-6} + C$$

Title	Integration Leading to Logarithmic Functions and Integration of Exponential Functions
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	21/12/2018

Applicable to

- JC H2 'A' Level Mathematics (Bridging only) and H1 'A' Level Mathematics
- Nanyang Polytechnic School of Chemical and Life Sciences – Mathematics for Life Sciences
- Nanyang Polytechnic School of Engineering – Engineering Mathematics 1B
- Institute of Technical Education – Calculus/Mathematics Modules
- Secondary School GCE 'O' Level Additional Mathematics

Function	Corresponding Integrals
$f(x) = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln x + C$
$f(x) = e^x$	$\int e^x dx = e^x + C$
$f(x) = ae^x$	$\int ae^x dx = ae^x + C$

(Generic Cases) Functions	Corresponding Integrals
$f(x) = \frac{1}{ax+b}$	$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln ax+b + C$
$f(x) = \frac{a}{x}$	$\int \frac{a}{x} = a(\ln(x)) + C$
$f(x) = e^{ax+b}$	$\int e^{ax+b} dx = \frac{1}{a} (e^{ax+b}) + C$

Find the following integrals

a) $\int \frac{3}{x} dx$

b) $\int e^{6x+5} dx$

c) $\int 5e^x dx$

d) $\int \frac{1}{9x+5} dx$

e) $\int e^{8x+6} dx$

Solutions

a) $\int \frac{3}{x} dx = 3(\ln(x)) = 3 \ln(x) + C$

b) $\int e^{6x+5} dx = \frac{1}{6}(e^{6x+5}) + C$

c) $\int 5e^x dx = 5e^x + C$

d) $\int \frac{1}{9x+5} dx = \frac{1}{9}(\ln(9x+5)) + C$

e) $\int e^{8x+6} dx = \frac{1}{8}(e^{8x+6}) + C$

Title	Additional Mathematics ['N' Levels] and Computing Mathematics – Finding the Definite Integral of a Function
Editor	Lee Jian Lian
Date	30/4/2018
Updates	(Removed Unnecessary Text From Original Version)

Disclaimer: This file does not cover logarithmic functions, exponential function and trigonometric function.

You need the following prerequisite knowledge before proceeding

- Able to find indefinite integral of a function
- Understand the notation of indefinite integral

Notation used in finding definite integral	
$\int_h^g f'(x) = f(g) - f(h) $	
<i>g and h</i>	Refers to the boundary between the two points on the x-axis you are finding the area under graph for
$ n $	This is sometimes referred to as the absolute value of n , any negative value that is between the two vertical strokes becomes positive while positive value within the vertical strokes remains positive.

Example 1

Find the following definite integral

$$\int_0^2 x^2 + 1 \, dx =$$

Indefinite integral of the mentioned function is:	$\int_0^2 x^2 + 1 \, dx = \frac{x^3}{3} + x$
Since we are finding the definite integral, we rewrite the integral in the following notation	$\left[\frac{x^3}{3} + x \right]_0^2$
Substitute 0 and 2 into the value x .	$\left \frac{2^3}{3} + 2 - \left(\frac{0^3}{3} + 0 \right) \right =$
Final Answer	$\frac{14}{3} \text{ units}$

Example 2

Find the following definite integral

$$\int_0^1 4 + 3x^2 \, dx$$

The indefinite integral of the function is	$\int_0^1 4 + 3x^2 = 4x + \frac{3x^3}{3}$
Rewrite using the following notation	$\left[4x + \frac{3x^3}{3} \right]_0^1$
Substitute 0 and 1 into the value of x	$\left 4(1) + \frac{3(1)^3}{3} - \left(4(0) + \frac{3(0)^3}{3} \right) \right $
Final Answer	5 units

Example 3

Find the following definite integral

$$\int_7^0 6x^7 + 2x - 6x^2 + 5$$

The indefinite integral of the function is	$\int_7^0 6x^7 + 2x - 6x^2 + 5 = \frac{6x^8}{8} + \frac{2x^2}{2} - 6\left(\frac{x^3}{3}\right) + 5x$
Rewrite using the following notation	$\left[\frac{6x^8}{8} + \frac{2x^2}{2} - 6\left(\frac{x^3}{3}\right) + 5x \right]_7^0$
Substitute 0 and 7 into the value of x	$\left \frac{6(0)^8}{8} + \frac{2(0)^2}{2} - 6\left(\frac{0^3}{3}\right) + 5(0) - \left[\frac{6(7)^8}{8} + \frac{2(7)^2}{2} - \frac{6(7^3)}{3} + 5(7) \right] \right $
Final Answer	$= 4322998.75 \text{ units}$

Example 4

Find the following definite integral

$$\int_{10}^2 \sqrt{x^3 + 2x - 9}$$

Indefinite integral of the function is	$\frac{3}{4}x^{\frac{4}{3}} + x^2 - 9x$
Rewrite using the notation as shown	$\left[\frac{3}{4}x^{\frac{4}{3}} + x^2 - 9x\right]_{10}^2$
Substitute 2 and 10 to find the definite integral	$\left \frac{3}{4}(2)^{\frac{4}{3}} + 2^2 - 9(2) - \left[\frac{3}{4}(10)^{\frac{4}{3}} + 10^2 - 9(10)\right]\right $
Answer is	38.268 <i>units</i>